A basis construction for the Shi arrangement of the type B_ℓ or C_ℓ

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Abstract

The Shi arrangement is an affine arrangement of hyperplanes consisting of the hyperplanes of the Weyl arrangement and their parallel translations. It was introduced by J.-Y. Shi in the study of the Kazhdan-Lusztig representation of the affine Weyl groups. M. Yoshinaga showed that the cone over every Shi arrangement is free. In this paper, we construct an explicit basis for the derivation module of the cone over the Shi arrangements of the type B_ℓ or C_ℓ.

Keywords: Hyperplane arrangement; Shi arrangement; Free arrangement; Derivations

1 Introduction

Let E be an ℓ-dimensional real Euclidean space. Let Φ be an irreducible root system and Φ_+ denote the set of positive roots of Φ. The **Weyl arrangement** of the type Φ is denoted by $\mathcal{A}(\Phi)$:

$$\mathcal{A}(\Phi) = \{H_\alpha \mid \alpha \in \Phi_+\}, \text{ where } H_\alpha = \{v \in E \mid \alpha(v) = 0\}.$$

Let

$$H_{\alpha,1} = \{v \in E \mid \alpha(v) = 1\}.$$

Then the **Shi arrangement** is given by

$$\mathcal{A}(\Phi) \cup \{H_{\alpha,1} \mid \alpha \in \Phi_+\} = \bigcup_{\alpha \in \Phi_+} \{H_\alpha, H_{\alpha,1}\}.$$

*Research Fellow of the Japan Society for the Promotion of Science.

Embed the ℓ-dimensional space E into $V = \mathbb{R}^{\ell+1}$ by adding a new coordinate z such that E is defined by the equation $z = 1$ in V. Then, as in [5, Definition 1.15], we have the **cone** $\mathcal{S}(\Phi)$ over the Shi arrangement. It is a central arrangement in V defined by

$$Q(\mathcal{S}(\Phi)) = z \prod_{\alpha \in \Phi_+} \alpha(\alpha - z) = 0.$$

Let S be the algebra of polynomial functions on V and let $\mathrm{Der}(S)$ be the module of derivations of S to itself

$$\mathrm{Der}(S) = \{\theta : S \to S \mid \theta \text{ is } \mathbb{R}\text{-linear and } \theta(fg) = f\theta(g) + g\theta(f) \text{ for any } f, g \in S\}.$$

The derivation module $D(\mathcal{S}(\Phi))$ is defined by

$$\begin{aligned} D(\mathcal{S}(\Phi)) = \{\theta \in \mathrm{Der}(S) \mid \theta(z) &\text{ is divisible by } z, \theta(\alpha) \text{ is divisible by } \alpha \\ &\text{and } \ \theta(\alpha - z) \text{ is divisible by } \alpha - z \text{ for any } \alpha \in \Phi_+\}. \end{aligned}$$

We say that $\mathcal{S}(\Phi)$ is **free** if $D(\mathcal{S}(\Phi))$ is a free S-module.

Let $x_1, \dots, x_\ell$ be an orthonormal basis for the dual space E^*. In this paper we explicitly choose root systems Φ^B and Φ^C, and positive root systems Φ_+^B and Φ_+^C of the types B_ℓ and C_ℓ respectively as follows:

$$\begin{aligned} \Phi^B &:= \{\pm x_i, \pm x_p \pm x_q \in E^* \mid 1 \le i \le \ell, 1 \le p < q \le \ell\}, \\ \Phi_+^B &:= \{x_i, x_p \pm x_q \in \Phi^B \mid 1 \le i \le \ell, 1 \le p < q \le \ell\}, \\ \Phi^C &:= \{\pm 2x_i, \pm x_p \pm x_q \in E^* \mid 1 \le i \le \ell, 1 \le p < q \le \ell\}, \\ \Phi_+^C &:= \{2x_i, x_p \pm x_q \in \Phi^C \mid 1 \le i \le \ell, 1 \le p < q \le \ell\}. \end{aligned}$$

We express the cones over the Shi arrangements of the types B_ℓ and C_ℓ by $\mathcal{S}(B_\ell)$ and $\mathcal{S}(C_\ell)$ respectively.

In the study of the Kazhdan-Lusztig representation theory of the affine Weyl groups, J.-Y. Shi introduced the Shi arrangements for the type A_ℓ in [6]. Later a good number of articles, including [1, 2, 4, 8, 10], study the Shi arrangements. M. Yoshinaga proved in [10] that the cone over the Shi arrangement is a free arrangement by settling the Edelman-Reiner conjecture in [2] which asserts that the generalized Shi and Catalan arrangements are free. However, even in the case of the cone over the Shi arrangement of the type A_ℓ, no basis was constructed explicitly at that time. Recently a basis for the cone over the Shi arrangement of the type A_ℓ is constructed explicitly in [9] and of the type D_ℓ in [3]. In those papers the most important ingredients of their recipes are the Bernoulli polynomials $B_k(x)$ and their

relatives $B_{r,s}(x)$. In the present paper, we construct bases for the cones over the Shi arrangements of the types B_ℓ and C_ℓ by using new Bernoulli-like polynomials $B^B_{r,s}(x)$ and $B^C_{r,s}(x)$.

The organization of this paper is as follows: in Section 2, we will construct ℓ derivations $\varphi^B_1, \dots, \varphi^B_\ell$ belonging to $D(\mathcal{S}(B_\ell))$. In Section 3, we will prove that they, together with the Euler derivation, form a basis of $D(\mathcal{S}(B_\ell))$. In Section 4, we present a similar construction of a basis for $D(\mathcal{S}(C_\ell))$ for the type C_ℓ.

2 A basis construction for the type B_ℓ

Definition 2.1. *For $(r, s) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{\geq 0}$, define a polynomial $B^B_{r,s}(x)$ in x satisfying the following two conditions:*

(i) $B^B_{r,s}(x+1) - B^B_{r,s}(x) = \dfrac{(x+1)^r - (-x)^r}{(x+1) - (-x)}(x+1)^s(-x)^s,$

(ii) $B^B_{r,s}(0) = 0.$

Note that $\frac{(x+1)^r-(-x)^r}{(x+1)-(-x)}$ is a polynomial either of degree $r-1$ (when r is odd) or of degree $r-2$ (when r is even). It is thus easy to see that $B^B_{r,s}(x)$ uniquely exists and

$$\deg B^B_{r,s}(x) = \begin{cases} r+2s & \text{if } r \text{ is odd,} \\ r+2s-1 & \text{if } r \text{ is even.} \end{cases}$$

Lemma 2.2. *$B^B_{r,s}(x)$ is an odd function.*

Proof. Replacing x with $-x-1$ in 2.1 (i), we have

$$\begin{aligned} B^B_{r,s}(-x) - B^B_{r,s}(-x-1) &= \frac{(-x)^r - (x+1)^r}{(-x) - (x+1)}(-x)^s(x+1)^s \\ &= B^B_{r,s}(x+1) - B^B_{r,s}(x). \end{aligned}$$

Then we get $F(x) = F(x+1)$ where $F(x) := B^B_{r,s}(x) + B^B_{r,s}(-x)$. Thus we obtain

$$F(n) = F(n-1) = \cdots = F(0) = 0 \ \ (n \in \mathbb{Z}_{\geq 0})$$

and

$$B^B_{r,s}(x) + B^B_{r,s}(-x) = F(x) = 0.$$

$\square$

Definition 2.3. *The homogenization* $\overline{B}^B_{r,s}(x,z)$ *of* $B^B_{r,s}(x)$ *is defined by*

$$\overline{B}^B_{r,s}(x,z) := z^{r+2s}B^B_{r,s}(x/z).$$

Let $1 \le j \le \ell$. Define

$$I_1^{(j)} = \{x_1, \dots, x_{j-1}\}, \;\; I_2^{(j)} = \{x_j\}, \;\; I_3^{(j)} = \{x_{j+1}, \dots, x_\ell\}$$

Let $\sigma_k(y_1, y_2, \dots)$ $(k \in \mathbb{Z}_{\ge 0})$ denote the elementary symmetric polynomials in $y_1, y_2, \dots$ of degree k. Then define

$$\sigma_k^{(2,j)} := \sigma_k(x_j), \;\; \tau_k^{(3,j)} := \sigma_k(x_{j+1}^2, \dots, x_\ell^2).$$

Definition 2.4. *Let* ∂_i $(1 \le i \le \ell)$ *and* ∂_z *denote* $\partial/\partial x_i$ *and* $\partial/\partial z$ *respectively. Define the Euler derivation*

$$\theta_E := z\partial_z + \sum_{i=1}^{\ell} x_i\partial_i$$

and the following homogeneous derivations

$$\varphi_j^B := (-1)^j \sum_{i=1}^{\ell} \Bigg\{ \sum_{\substack{N_1, N_2 \subset I_1^{(j)} \\ N_1 \cap N_2 = \emptyset}} \Bigg(\prod_{x_t \in N_1} x_t^2 \Bigg) \Bigg(\prod_{x_t \in N_2} (-x_t z) \Bigg)$$
$$\sum_{\substack{0 \le k_2 \le 1 \\ 0 \le k_3 \le \ell - j}} (-1)^{k_2+k_3} \sigma_{k_2}^{(2,j)} \tau_{k_3}^{(3,j)} \; \overline{B}^B_{r,s}(x_i, z) \Bigg\} \partial_i,$$

where

$$r := 2\ell - 2j - k_2 - 2k_3 + 2 \ge 1, \quad s := |I_1^{(j)} \setminus (N_1 \cup N_2)| = (j-1) - |N_1| - |N_2| \ge 0$$

for $1 \le j \le \ell$.

It is easy to see that each φ_j^B is homogeneous derivation of degree 2ℓ which is equal to the Coxeter number for B_ℓ. We will prove that the derivations θ_E and $\varphi_1^B, \dots, \varphi_\ell^B$ form a basis for $D(\mathcal{S}(B_\ell))$. First we will verify the following

Proposition 2.5. *Let* $\varepsilon \in \{-1, 0, 1\}$. *Then we have the following congruence relations:*

$$\overline{B}^B_{r,s}(x_p, z) + \varepsilon \overline{B}^B_{r,s}(x_q, z) \equiv 0 \mod (x_p + \varepsilon x_q),$$

$$\overline{B}^B_{r,s}(x_p, z) + \varepsilon \overline{B}^B_{r,s}(x_q, z) \equiv (x_p + \varepsilon x_q) \frac{x_p^r - (\varepsilon x_q)^r}{x_p - \varepsilon x_q} (x_p \cdot \varepsilon x_q)^s \mod (x_p + \varepsilon x_q - z).$$

Proof. The first congruence follows from Definition 2.1 (ii) and Lemma 2.2. Let the congruent notation $\equiv$ in the following calculation be modulo the ideal $(x_p + \varepsilon x_q - z)$. By Definition 2.1 and Lemma 2.2, we have

$$\begin{aligned}
&\overline{B}^B_{r,s}(x_p, z) + \varepsilon \overline{B}^B_{r,s}(x_q, z) = \overline{B}^B_{r,s}(x_p, z) + \overline{B}^B_{r,s}(\varepsilon x_q, z) \\
&= z^{r+2s}\{B^B_{r,s}\left(\frac{x_p}{z}\right) + B^B_{r,s}\left(\frac{\varepsilon x_q}{z}\right)\} \\
&\equiv (x_p + \varepsilon x_q)^{r+2s}\left\{B^B_{r,s}\left(\frac{x_p}{x_p + \varepsilon x_q}\right) + B^B_{r,s}\left(\frac{\varepsilon x_q}{x_p + \varepsilon x_q}\right)\right\} \\
&= (x_p + \varepsilon x_q)^{r+2s}\left\{B^B_{r,s}\left(\frac{x_p}{x_p + \varepsilon x_q}\right) - B^B_{r,s}\left(-\frac{\varepsilon x_q}{x_p + \varepsilon x_q}\right)\right\} \\
&= (x_p + \varepsilon x_q)^{r+2s}\frac{\left(\frac{x_p}{x_p+\varepsilon x_q}\right)^r - \left(\frac{\varepsilon x_q}{x_p+\varepsilon x_q}\right)^r}{\frac{x_p}{x_p+\varepsilon x_q} - \frac{\varepsilon x_q}{x_p+\varepsilon x_q}}\left(\frac{x_p}{x_p + \varepsilon x_q}\right)^s\left(\frac{\varepsilon x_q}{x_p + \varepsilon x_q}\right)^s \\
&= (x_p + \varepsilon x_q)\frac{x_p^r - (\varepsilon x_q)^r}{x_p - \varepsilon x_q}(x_p \cdot \varepsilon x_q)^s.
\end{aligned}$$

$\square$

Proposition 2.6. *The derivations φ^B_j $(1 \le j \le \ell)$ belong to the module $D(\mathcal{S}(B_\ell))$.*

Proof. By Proposition 2.5, we first have

$$\begin{aligned}
\varphi^B_j(x_p + \varepsilon x_q) = (-1)^j &\sum_{\substack{N_1, N_2 \subset I^{(j)}_1 \\ N_1 \cap N_2 = \emptyset}} \left(\prod_{x_t \in N_1} x_t^2\right)\left(\prod_{x_t \in N_2}(-x_t z)\right) \\
&\sum_{\substack{0 \le k_2 \le 1 \\ 0 \le k_3 \le \ell - j}} (-1)^{k_2+k_3}\sigma^{(2,j)}_{k_2}\tau^{(3,j)}_{k_3}(\overline{B}^B_{r,s}(x_p, z) + \varepsilon \overline{B}^B_{r,s}(x_q, z)) \\
&\equiv 0 \mod (x_p + \varepsilon x_q)
\end{aligned}$$

for $1 \le j \le \ell$. Thus we conclude that $\varphi^B_j(x_p), \varphi^B_j(x_p \pm x_q)$ are divisible by $x_p, x_p \pm x_q$ for $1 \le p \le \ell, 1 \le p < q \le \ell$ respectively.

Let the congruent notation $\equiv$ in the following calculation be modulo the

ideal $(x_p+\varepsilon x_q-z)$. By Proposition 2.5, for $1\le j\le \ell$, we also have

$$
\begin{aligned}
&\varphi_j^B(x_p+\varepsilon x_q-z)=\varphi_j^B(x_p+\varepsilon x_q)\\
&=(-1)^j\sum_{\substack{N_1,N_2\subset I_1^{(j)}\\ N_1\cap N_2=\emptyset}}\Big(\prod_{x_t\in N_1}x_t^2\Big)\Big(\prod_{x_t\in N_2}(-x_tz)\Big)\\
&\qquad\sum_{\substack{0\le k_2\le 1\\ 0\le k_3\le \ell-j}}(-1)^{k_2+k_3}\sigma_{k_2}^{(2,j)}\tau_{k_3}^{(3,j)}(\overline{B}_{r,s}^B(x_p,z)+\varepsilon\overline{B}_{r,s}^B(x_q,z))\\
&\equiv(-1)^j\sum_{\substack{N_1,N_2\subset I_1^{(j)}\\ N_1\cap N_2=\emptyset}}\Big(\prod_{x_t\in N_1}x_t^2\Big)\Big(\prod_{x_t\in N_2}(-x_t(x_p+\varepsilon x_q))\Big)\\
&\qquad\sum_{\substack{0\le k_2\le 1\\ 0\le k_3\le \ell-j}}(-1)^{k_2+k_3}\sigma_{k_2}^{(2,j)}\tau_{k_3}^{(3,j)}(x_p+\varepsilon x_q)\frac{x_p^r-(\varepsilon x_q)^r}{x_p-\varepsilon x_q}(x_p\cdot\varepsilon x_q)^s\\
&=(x_p+\varepsilon x_q)\sum_{\substack{N_1,N_2\subset I_1^{(j)}\\ N_1\cap N_2=\emptyset}}\Big(\prod_{x_t\in N_1}x_t^2\Big)\Big(\prod_{x_t\in N_2}(-x_t(x_p+\varepsilon x_q))\Big)(x_p\cdot\varepsilon x_q)^s\\
&\qquad\frac{(-1)^{\ell+1}}{x_p-\varepsilon x_q}\Big\{\sum_{\substack{0\le k_2\le 1\\ 0\le k_3\le \ell-j}}(-1)^{\ell-j+1-k_2-k_3}\sigma_{k_2}^{(2,j)}\tau_{k_3}^{(3,j)}x_p^r\\
&\qquad\qquad-\sum_{\substack{0\le k_2\le 1\\ 0\le k_3\le \ell-j}}(-1)^{\ell-j+1-k_2-k_3}\sigma_{k_2}^{(2,j)}\tau_{k_3}^{(3,j)}(\varepsilon x_q)^r\Big\}.
\end{aligned}
$$

Here,

$$
\begin{aligned}
&\sum_{\substack{N_1,N_2\subset I_1^{(j)}\\ N_1\cap N_2=\emptyset}}\Big(\prod_{x_t\in N_1}x_t^2\Big)\Big(\prod_{x_t\in N_2}(-x_t(x_p+\varepsilon x_q))\Big)(x_p\cdot\varepsilon x_q)^s\\
&=\prod_{t=1}^{j-1}(x_t^2-(x_p+\varepsilon x_q)x_t+x_p\cdot\varepsilon x_q)=\prod_{t=1}^{j-1}(x_t-x_p)(x_t-\varepsilon x_q)
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{\substack{0\le k_2\le 1\\ 0\le k_3\le \ell-j}}(-1)^{\ell-j+1-k_2-k_3}\sigma_{k_2}^{(2,j)}\tau_{k_3}^{(3,j)}x_p^r\\
&=x_p\sum_{k_2=0}^{1}\sigma_{k_2}^{(2,j)}(-x_p)^{1-k_2}\sum_{k_3=0}^{\ell-j}\tau_{k_3}^{(3,j)}(-x_p^2)^{\ell-j-k_3}=x_p(x_j-x_p)\prod_{t=j+1}^{\ell}(x_t^2-x_p^2).
\end{aligned}
$$

If $1 \le p \le j-1$, then

$$\prod_{t=1}^{j-1}(x_t - x_p)(x_t - \varepsilon x_q) = 0.$$

If $j \le p < q \le \ell$, then

$$x_p(x_j - x_p)\left(\prod_{t=j+1}^{\ell}(x_t^2 - x_p^2)\right) = \varepsilon x_q(x_j - \varepsilon x_q)\left(\prod_{t=j+1}^{\ell}(x_t^2 - (\varepsilon x_q)^2)\right) = 0.$$

Therefore

$$\begin{aligned}
&\varphi_j^B(x_p + \varepsilon x_q - z) \\
&\equiv (-1)^{\ell+1}\frac{x_p + \varepsilon x_q}{x_p - \varepsilon x_q}\prod_{t=1}^{j-1}(x_t - x_p)(x_t - \varepsilon x_q) \\
&\qquad \left\{x_p(x_j - x_p)\left(\prod_{t=j+1}^{\ell}(x_t^2 - x_p^2)\right) - \varepsilon x_q(x_j - \varepsilon x_q)\left(\prod_{t=j+1}^{\ell}(x_t^2 - (\varepsilon x_q)^2)\right)\right\} \\
&= 0
\end{aligned}$$

for all pairs (p,q) with $1 \le p < q \le \ell$ and $\varepsilon \in \{-1, 0, 1\}$. Hence $\varphi_j^B \in D(\mathcal{S}(B_\ell))$ for $1 \le j \le \ell$. $\square$

3 The W-equivariance

Recall that $\mathcal{A}(\Phi)$ is the Weyl arrangement in E corresponding to the irreducible root system Φ. Then we may identify

$$\overline{S} := S/zS \simeq \mathbb{R}[x_1, \ldots, x_\ell]$$

with the algebra of polynomial functions on E. In [7] L. Solomon and H. Terao studied the $\overline{S}$-module

$$D(\mathcal{A}(\Phi), 2) := \{\theta \in \mathrm{Der}(\overline{S}) \mid \theta(\alpha_H) \in \overline{S}\alpha_H^2, H \in \mathcal{A}(\Phi)\},$$

which was denoted by $E(\mathcal{A})$ in [7]. Let h be the Coxeter number for Φ. Define

$$D(\mathcal{A}(\Phi), 2)_h := \{\theta \in D(\mathcal{A}(\Phi), 2) \mid \deg\theta = h\} \cup \{0\},$$

which is a real vector space. Note that the Weyl group W corresponding to Φ naturally acts on $D(\mathcal{A}(\Phi), 2)$ and $D(\mathcal{A}(\Phi), 2)_h$. We also define an S-submodule

$$D_0(\mathcal{S}(\Phi)) := \{\varphi \in D(\mathcal{S}(\Phi)) \mid \varphi(z) = 0\}$$

of $D(\mathcal{S}(\Phi))$. Then $D(\mathcal{S}(\Phi))$ has a decomposition

$$D(\mathcal{S}(\Phi)) = D_0(\mathcal{S}(\Phi)) \oplus S\theta_E$$

over S. Let

$$D_0(\mathcal{S}(\Phi))_h := \{\varphi \in D_0(\mathcal{S}(\Phi)) \mid \deg \varphi = h\} \cup \{0\},$$

which is a real vector space. If $\varphi \in D_0(\mathcal{S}(\Phi))$, then $\varphi(\alpha_H) \in \alpha_H(\alpha_H - z)S$ for any $H \in \mathcal{A}(\Phi)$. Let $\overline{\varphi} := \varphi|_{z=0}$ be the restriction of φ to $z = 0$. Then $\overline{\varphi}(\alpha_H) \in \alpha_H^2 \overline{S}$ for any $H \in \mathcal{A}(\Phi)$, hence $\overline{\varphi} \in D(\mathcal{A}(\Phi), 2)$.

Theorem 3.1. *(1) (L. Solomon-H. Terao[7]) The $\overline{S}$-module $D(\mathcal{A}(\Phi), 2)$ is a free module with a basis consisting of ℓ derivations homogeneous of degree h. In other words, we have an isomorphism*

$$D(\mathcal{A}(\Phi), 2) \simeq D(\mathcal{A}(\Phi), 2)_h \otimes_{\mathbb{R}} \overline{S}.$$

(2) (M. Yoshinaga[10]) The S-module $D_0(\mathcal{S}(\Phi))$ is a free module with a basis consisting of ℓ derivations homogeneous of degree h. In other words, we have an isomorphism

$$D_0(\mathcal{S}(\Phi)) \simeq D(\mathcal{S}(\Phi))_h \otimes_{\mathbb{R}} S.$$

Also the restriction map

$$\rho : D_0(\mathcal{S}(\Phi))_h \longrightarrow D(\mathcal{A}(\Phi), 2)_h$$

defined by $\varphi \mapsto \overline{\varphi} = \varphi|_{z=0}$ is a linear isomorphism.

Suppose that Φ is of the type B_ℓ in the rest of this section. Then we may define an explicit $\mathbb{R}$-linear map

$$\Psi : E^* \to D_0(\mathcal{S}(B_\ell))_h$$

by

$$\Psi(x_j) = \varphi_j^B \quad (1 \le j \le \ell)$$

using the derivations $\varphi_1^B, \dots, \varphi_\ell^B$ in Definition 2.4.

Theorem 3.2. *Let Φ be a root system of the type B_ℓ.*

(1) The map

$$\Xi : E^* \to D(\mathcal{A}(B_\ell), 2)_h$$

defined by $\Xi = \rho \circ \Psi$ is a W-equivariant isomorphism.

(2) The map

$$\Psi : E^* \to D_0(\mathcal{S}(B_\ell))_h$$

is a linear isomorphism.

Proof. (1) Since

$$\overline{B}^B_{r,s}(x_i,0) = \begin{cases} (-1)^s x_i^{r+2s}/(r+2s) & (r : \text{odd number}) \\ 0 & (r : \text{even number}) \end{cases},$$

$$\begin{aligned}
\Xi(x_j)(x_i) &= (\rho \circ \Psi(x_j))(x_i) = \varphi_j^B(x_i)|_{z=0} \\
&= (-1)^j x_j \sum_{N_1 \subset I_1^{(j)}} \left(\prod_{x_t \in N_1} x_t^2 \right) \sum_{k_3=0}^{\ell-j} (-1)^{1+k_3} \tau_{k_3}^{(3,j)} (-1)^s \frac{x_i^{r+2s}}{r+2s} \\
&= (-1)^j x_j \sum_{m=0}^{j-1} \sum_{\substack{N_1 \subset I_1^{(j)} \\ |N_1| = m}} \left(\prod_{x_t \in N_1} x_t^2 \right) \sum_{k_3=0}^{\ell-j} (-1)^{1+k_3} \tau_{k_3}^{(3,j)} (-1)^s \frac{x_i^{r+2s}}{r+2s} \\
&= x_j \sum_{m=0}^{j-1} \tau_m^{(1,j)} \sum_{k_3=0}^{\ell-j} (-1)^{m+k_3} \tau_{k_3}^{(3,j)} \frac{x_i^{2\ell-2m-2k_3-1}}{2\ell-2m-2k_3-1} \\
&= x_j \sum_{k=0}^{\ell-1} (-1)^k \sigma_k(x_1^2, \dots, x_{j-1}^2, x_{j+1}^2, \dots, x_\ell^2) \frac{x_i^{2\ell-2k-1}}{2\ell-2k-1}.
\end{aligned}$$

Thus we obtain

$$\Xi(x_j) = x_j \sum_{k=0}^{\ell-1} (-1)^k \sigma_k(x_1^2, \dots, x_{j-1}^2, x_{j+1}^2, \dots, x_\ell^2) \sum_{i=1}^{\ell} \left(\frac{x_i^{2\ell-2k-1}}{2\ell-2k-1} \right) \partial_i.$$

Since

$$\sum_{i=1}^{\ell} \left(\frac{x_i^{2\ell-2k-1}}{2\ell-2k-1} \right) \partial_i$$

is a W-invariant derivation and the correspondence

$$x_j \mapsto x_j \sigma_k(x_1^2, \dots, x_{j-1}^2, x_{j+1}^2, \dots, x_\ell^2) \quad (0 \le k \le \ell - 1)$$

is W-equivariant for every $k \in \mathbb{Z}_{\ge 0}$, we conclude that Ξ is W-equivariant. Therefore Ξ is bijective by Schur's lemma.

(2) follows from (1) because the restriction map ρ is bijective by Theorem 3.1 (2). □

Theorem 3.3. *The derivations $\theta_E, \varphi_1^B, \dots, \varphi_\ell^B$ form a basis for $D(\mathcal{S}(B_\ell))$.*

Proof. It is enough to show that $\varphi_1^B, \dots, \varphi_\ell^B$ form a basis for $D_0(\mathcal{S}(B_\ell))$. Recall that each $\Psi(x_j) = \varphi_j^B$ belongs to $D_0(\mathcal{S}(B_\ell))_h$. Theorems 3.1 (2) and 3.2 (2) complete the proof. □

4 A basis construction for the type C_ℓ

Definition 4.1. *For $(r, s) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{\geq 0}$, define a polynomial $B^C_{r,s}(x)$ in x satisfying the following two conditions:*

(i) $B^C_{r,s}(x+1) - B^C_{r,s}(x) = \{(x+1)^{r-1} + (-x)^{r-1}\}(x+1)^s(-x)^s$,

(ii) $B^C_{r,s}(0) = 0$.

It is easy to see that $B^C_{r,s}(x)$ uniquely exists and

$$\deg B^C_{r,s}(x) = \begin{cases} r + 2s & \text{if } r \text{ is odd,} \\ r + 2s - 1 & \text{if } r \text{ is even.} \end{cases}$$

The following lemma can be proved by a smilar argument to the proof of Lemma 2.2:

Lemma 4.2. *$B^C_{r,s}(x)$ is an odd function.*

Definition 4.3. *The homogenization $\overline{B}^C_{r,s}(x, z)$ of $B^C_{r,s}(x)$ is defined by*

$$\overline{B}^C_{r,s}(x, z) := z^{r+2s} B^C_{r,s}(x/z).$$

Definition 4.4. *Define homogeneous derivations*

$$\varphi^C_j := (-1)^j \sum_{i=1}^{\ell} \Bigg\{ \sum_{\substack{N_1, N_2 \subset I_1^{(j)} \\ N_1 \cap N_2 = \emptyset}} \Bigg(\prod_{x_t \in N_1} x_t^2 \Bigg) \Bigg(\prod_{x_t \in N_2} (-x_t z) \Bigg) \sum_{\substack{0 \leq k_2 \leq 1 \\ 0 \leq k_3 \leq \ell - j}} (-1)^{k_2+k_3} \sigma^{(2,j)}_{k_2} \tau^{(3,j)}_{k_3} \ \overline{B}^C_{r,s}(x_i, z) \Bigg\} \partial_i$$

where

$$r := 2\ell - 2j - k_2 - 2k_3 + 2 \geq 1, \quad s := |I_1^{(j)} \setminus (N_1 \cup N_2)| = (j-1) - |N_1| - |N_2| \geq 0$$

for $1 \leq j \leq \ell$.

Note that φ^C_j is exactly the same as φ^B_j with only one exception: the use of $B^C_{r,s}(x_i, z)$ instead of $B^B_{r,s}(x_i, z)$. Thus each φ^B_j is homogeneous derivation of degree 2ℓ which is equal to the Coxeter number for C_ℓ. We will prove that the derivations θ_E and $\varphi^C_1, \dots, \varphi^C_\ell$ form a basis for $D(\mathcal{S}(C_\ell))$. We first have the following Proposition:

Proposition 4.5. *Let $\varepsilon \in \{-1, 0, 1\}$. Then we have the following conguruence relations:*

$$\overline{B}^C_{r,s}(x_p, z) + \varepsilon \overline{B}^C_{r,s}(x_q, z) \equiv 0 \mod (x_p + \varepsilon x_q),$$

$$\overline{B}^C_{r,s}(x_p, z) + \varepsilon \overline{B}^C_{r,s}(x_q, z) \equiv (x_p + \varepsilon x_q)\{x_p^{r-1} + (\varepsilon x_q)^{r-1}\}(x_p \cdot \varepsilon x_q)^s \\ \mod (x_p + \varepsilon x_q - z).$$

Proof. Imitate the proof of Proposition 2.5. □

Proposition 4.6. *The derivations φ^C_j $(1 \leq j \leq \ell)$ belong to the module $D(\mathcal{S}(C_\ell))$.*

Proof. By Proposition 4.5, we first have

$$\begin{aligned}
&\varphi^C_j(x_p + \varepsilon x_q) \\
&= (-1)^j \sum_{\substack{N_1, N_2 \subset I_1^{(j)} \\ N_1 \cap N_2 = \emptyset}} \left(\prod_{x_t \in N_1} x_t^2\right)\left(\prod_{x_t \in N_2} (-x_t z)\right) \\
&\qquad \sum_{\substack{0 \leq k_2 \leq 1 \\ 0 \leq k_3 \leq \ell - j}} (-1)^{k_2 + k_3} \sigma^{(2,j)}_{k_2} \tau^{(3,j)}_{k_3} (\overline{B}^C_{r,s}(x_p, z) + \varepsilon \overline{B}^C_{r,s}(x_q, z)) \\
&\equiv 0 \qquad (\mathrm{mod}\ (x_p + \varepsilon x_q))
\end{aligned}$$

for $1 \leq j \leq \ell$. Thus we conclude that $\varphi^C_j(2x_p), \varphi^C_j(x_p \pm x_q)$ are divisible by $2x_p, x_p \pm x_q$ for $1 \leq p \leq \ell, 1 \leq p < q \leq \ell$ respectively.

Let the congruent notation $\equiv$ in the following calculation be modulo the ideal $(x_p + \varepsilon x_q - z)$. By Proposition 4.5, for $1 \leq j \leq \ell$, we also have

$$\begin{aligned}
&\varphi^C_j(x_p + \varepsilon x_q - z) = \varphi^C_j(x_p + \varepsilon x_q) \\
&= (-1)^j \sum_{\substack{N_1, N_2 \subset I_1^{(j)} \\ N_1 \cap N_2 = \emptyset}} \left(\prod_{x_t \in N_1} x_t^2\right)\left(\prod_{x_t \in N_2} (-x_t z)\right) \\
&\qquad \sum_{\substack{0 \leq k_2 \leq 1 \\ 0 \leq k_3 \leq \ell - j}} (-1)^{k_2 + k_3} \sigma^{(2,j)}_{k_2} \tau^{(3,j)}_{k_3} (\overline{B}^C_{r,s}(x_p, z) + \varepsilon \overline{B}^C_{r,s}(x_q, z))
\end{aligned}$$

$$\equiv (-1)^j \sum_{\substack{N_1,N_2\subset I_1^{(j)}\\ N_1\cap N_2=\emptyset}} \Bigg(\prod_{x_t\in N_1} x_t^2\Bigg)\Bigg(\prod_{x_t\in N_2}(-x_t(x_p+\varepsilon x_q))\Bigg)$$

$$\sum_{\substack{0\le k_2\le 1\\ 0\le k_3\le \ell-j}} (-1)^{k_2+k_3}\sigma_{k_2}^{(2,j)}\tau_{k_3}^{(3,j)}(x_p+\varepsilon x_q)\{x_p^{r-1}+(\varepsilon x_q)^{r-1}\}(x_p\cdot\varepsilon x_q)^s$$

$$= (x_p+\varepsilon x_q) \sum_{\substack{N_1,N_2\subset I_1^{(j)}\\ N_1\cap N_2=\emptyset}} \Bigg(\prod_{x_t\in N_1} x_t^2\Bigg)\Bigg(\prod_{x_t\in N_2}(-x_t(x_p+\varepsilon x_q))\Bigg)(x_p\cdot\varepsilon x_q)^s$$

$$(-1)^{\ell+1}\Bigg\{\sum_{\substack{0\le k_2\le 1\\ 0\le k_3\le \ell-j}} (-1)^{\ell-j+1-k_2-k_3}\sigma_{k_2}^{(2,j)}\tau_{k_3}^{(3,j)}x_p^{r-1}$$

$$+\sum_{\substack{0\le k_2\le 1\\ 0\le k_3\le \ell-j}} (-1)^{\ell-j+1-k_2-k_3}\sigma_{k_2}^{(2,j)}\tau_{k_3}^{(3,j)}(\varepsilon x_q)^{r-1}\Bigg\}.$$

Here,

$$\sum_{\substack{N_1,N_2\subset I_1^{(j)}\\ N_1\cap N_2=\emptyset}} \Bigg(\prod_{x_t\in N_1} x_t^2\Bigg)\Bigg(\prod_{x_t\in N_2}(-x_t(x_p+\varepsilon x_q))\Bigg)(x_p\cdot\varepsilon x_q)^s$$

$$= \prod_{t=1}^{j-1}(x_t^2-(x_p+\varepsilon x_q)x_t+x_p\cdot\varepsilon x_q) = \prod_{t=1}^{j-1}(x_t-x_p)(x_t-\varepsilon x_q),$$

and

$$\sum_{\substack{0\le k_2\le 1\\ 0\le k_3\le \ell-j}} (-1)^{\ell-j+1-k_2-k_3}\sigma_{k_2}^{(2,j)}\tau_{k_3}^{(3,j)}x_p^{r-1}$$

$$= \sum_{k_2=0}^{1}\sigma_{k_2}^{(2,j)}(-x_p)^{1-k_2}\sum_{k_3=0}^{\ell-j}\tau_{k_3}^{(3,j)}(-x_p^2)^{\ell-j-k_3} = (x_j-x_p)\prod_{t=j+1}^{\ell}(x_t^2-x_p^2).$$

If $1\le p\le j-1$, then

$$\prod_{t=1}^{j-1}(x_t-x_p)(x_t-\varepsilon x_q)=0.$$

If $j\le p<q\le \ell$ and $\varepsilon\in\{-1,1\}$, then

$$(x_j-x_p)\prod_{t=j+1}^{\ell}(x_t^2-x_p^2) = (x_j-\varepsilon x_q)\prod_{t=j+1}^{\ell}(x_t^2-(\varepsilon x_q)^2)=0.$$

Therefore

$$
\begin{aligned}
&\varphi_j^C(x_p+\varepsilon x_q - z)\\
&\equiv (-1)^{\ell-j+1}(x_p+\varepsilon x_q)\prod_{t=1}^{j-1}(x_t-x_p)(x_t-\varepsilon x_q)\\
&\qquad \left\{(x_j-x_p)\left(\prod_{t=j+1}^{\ell}(x_t^2-x_p^2)\right)+(x_j-\varepsilon x_q)\left(\prod_{t=j+1}^{\ell}(x_t^2-(\varepsilon x_q)^2)\right)\right\}\\
&=0
\end{aligned}
$$

for all pairs (p,q) with $1 \le p < q \le \ell$ where $\varepsilon \in \{-1,1\}$. When $p=q, \varepsilon=1$,

$$
\begin{aligned}
&\varphi_j^C(x_p+\varepsilon x_q - z) = \varphi_j^C(2x_p - z)\\
&\equiv (-1)^{\ell-j+1}(2x_p)\prod_{t=1}^{j-1}(x_t-x_p)^2\left\{2(x_j-x_p)\prod_{t=j+1}^{\ell}(x_t^2-x_p^2)\right\}\\
&=0
\end{aligned}
$$

for $1 \le p \le \ell$. Hence $\varphi_j \in D(\mathcal{S}(C_\ell))$ for $1 \le j \le \ell$. □

We may define an explicit $\mathbb{R}$-linear map

$$\Psi : E^* \to D_0(\mathcal{S}(C_\ell))_h$$

by

$$\Psi(x_j) = \varphi_j^C \quad (1 \le j \le \ell)$$

using the derivations $\varphi_1^C, \dots, \varphi_\ell^C$ in Definition 4.4.

Theorem 4.7. *Let Φ be a root system of the type C_ℓ.*

(1) The map

$$\Xi : E^* \to D(\mathcal{A}(C_\ell), 2)_h$$

defined by $\Xi = \rho \circ \Psi$ is a W-equivariant isomorphism.

(2) The map

$$\Psi : E^* \to D_0(\mathcal{S}(C_\ell))_h$$

is a linear isomorphism.

Proof. Since

$$\overline{B}^C_{r,s}(x_i,0) = 2\overline{B}^B_{r,s}(x_i,0) = \begin{cases}(-1)^s 2x_i^{r+2s}/(r+2s) & (r : \text{odd number})\\ 0 & (r : \text{even number})\end{cases},$$

we may prove this theorem in the same way as Theorem 3.2. □

Theorem 4.8. *The derivations $\theta_E, \varphi_1^C, \ldots, \varphi_\ell^C$ form a basis for $D(\mathcal{S}(C_\ell))$.*

Proof. Apply Theorems 4.7 (2) and and 3.1 (2) in the same way as the proof of Theorem 3.3. $\square$

Remark 4.9. *Since the W-equivariant isomorphism $\Xi : E^* \to D(\mathcal{A}(B_\ell), 2)_h$ in Theorem 3.2 (1) is unique up to a nonzero constant multiple by Schur's lemma, the derivations $\varphi_1^B|_{z=0}, \ldots, \varphi_\ell^B|_{z=0}$ coincide with the Solomon-Terao basis in [7] up to a nonzero constant multiple. Therefore, our construction of $\varphi_1^B, \ldots, \varphi_\ell^B$ can be regarded as an explicit realization of the basis existence theorem by M. Yoshinaga in [10]. This is also true for the type C_ℓ.*

Acknowledgment. The author is deeply grateful to Professor H. Terao for his advice and support.

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